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1986 J. Phys. A: Math. Gen. 19 91

(http://iopscience.iop.org/0305-4470/19/1/013)

J. Phys. A: Math. Gen. 19 (1986) 91-105. Printed in Great Britain

Quantum mechanics in coherent algebras on phase space

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Received 30 August 1983, in final form 29 May 1985

Abstract. Quantum mechanics is formulated on a quantum mechanical phase space. The algebra of observables and states is represented by an algebra of functions on phase space that fulfills a certain coherence condition, expressing the quantum mechanical superposition principle. The trace operation is an integration over phase space. In the case where the canonical variables independently run from $-\infty$ to $+\infty$ the formalism reduces to the representation of quantum mechanics by Wigner distributions. However, the notion of coherent algebras allows to apply the formalism to spaces for which the Wigner mapping is not known. Quantum mechanics of a particle in a plane in polar coordinates is discussed as an example.

1. Introduction

The desire to describe quantum mechanics on phase space is almost as old as quantum mechanics itself. First steps in this direction were made by Wigner [1] and later Groenewold [2], Moyal [3] and others have contributed to the subject. References [4-6] summarise the state of this development. Wigner constructed a mapping that allows us to map the structure of Hilbert space quantum mechanics on a corresponding structure on phase space. The inverse mapping was given by Weyl in an earlier work [7] without a discussion of quantum mechanics on phase space. However, this mapping is only known for the case where the canonical variables independently run from $-\infty$ to $+\infty$.

In this paper a formulation of quantum mechanics on phase space is presented that does not start from a relation to a Hilbert space formulation. Therefore our formalism easily extends to phase spaces with more complicated topologies, e.g. a cylinder, for which a Wigner mapping is not known. On the other hand, once we have a quantum mechanics on a phase space with a given topology, a relation to a Hilbert space formalism is obvious. Thus, afterwards one gets Wigner mappings for complicated topologies almost automatically.

Instead of representing states and observables by operators on Hilbert space we represent them by functions on a phase space. We define a non-commutative product for these functions so that we can find the algebra of quantum mechanics among functions on phase space. In our formalism this algebra is realised by a set of functions that fulfills a certain coherence condition. This condition corresponds to the superposition principle in Hilbert space quantum mechanics. The notion of coherent algebras will be decisive in extending the theory to complicated phase spaces.

In § 2 we introduce an associative but non-commutative product based on exponentiation of the Poisson operator. Section 3 shows how quantum mechanics on phase space can be guessed from classical mechanics. Thereby coherent algebras are

0305-4470/86/010091+15$02.50

introduced. Section 4 shows that our result in $\mathbb{R}^{2n}$ is equivalent to Weyl-Moyal quantisation. For identical bosons we discuss the 'reduction of phase volume' from the coherent algebra point of view. In § 5 we apply the technique of coherent algebras to formulate quantum mechanics of a particle in a plane using polar coordinates as canonical variables.

2. The * product

Let q_i, p_i; $i = 1, \ldots, n$ be cartesian coordinates of the $2n$-dimensional space $\mathbb{R}^{2n}$. The differential operator

$$\mathscr{P} = \sum_{i=1}^{n} \left(\frac{\overleftarrow{\partial}}{\partial q_i} \frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial q_i} \right) \tag{1}$$

can be used to write a Poisson bracket if one identifies the q's and p's with canonical coordinates of a mechanical system

$$\{A, B\} = A\mathscr{P}B. \tag{2}$$

We define the kth power of $\mathscr{P}$ as follows

$$\mathscr{P}^k = \sum_{\substack{i_1 \ldots i_k \\ j_1 \ldots j_k}} \varepsilon_{i_1 j_1} \ldots \varepsilon_{i_k j_k} \frac{\overleftarrow{\partial}}{\partial x_{i_1}} \ldots \frac{\overleftarrow{\partial}}{\partial x_{i_k}} \frac{\overrightarrow{\partial}}{\partial x_{j_k}} \ldots \frac{\overrightarrow{\partial}}{\partial x_{j_1}} \tag{3}$$

where $x_i = q_i$; $x_{i+n} = p_i$; $i \leq n$ and $\varepsilon_{ij} = \delta_{i(j-n)} - \delta_{i(j+n)}$. The structure $\langle \mathbb{R}^{2n}, \mathscr{P} \rangle$ can be identified with a classical phase space. On the other hand, the space of coordinates q_i, p_i together with the entirety of all powers $\mathscr{P}, \mathscr{P}^2, \ldots$ i.e. the structure $\langle \mathbb{R}^{2n}, \mathscr{P}, \mathscr{P}^2, \ldots \rangle$ does not permit such an interpretation. $\mathscr{P}$ is invariant under canonical transformations whereas $\mathscr{P}^k$, $k > 1$ is not unless the transformation is linear. The powers of $\mathscr{P}$ serve to define the following product of functions on $\mathbb{R}^{2n}$ (cf [2]):

$$* = \exp(\tfrac{1}{2} i\hbar \mathscr{P}) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \mathscr{P} \right)^k . \tag{4}$$

Two functions A, B are multiplied by applying the * operator between A and B; $A * B$. We will call the structure $\langle \mathbb{R}^{2n}, * \rangle$ the quantum mechanical phase space. In this work we will not define the class of functions that can be multiplied with this * product. We assume analytic properties and occasionally we will also multiply some distributions with due care.

Despite the fact that $\mathscr{P}$ only satisfies a Jacobi identity the * product turns out to be associative:

$$A * (B * C) = (A * B) * C. \tag{5}$$

This identity has been proved by Plebański [8] in a way that is independent of any relation to corresponding Hilbert space structures.

The * product is not commutative. The odd-power contributions of $(\tfrac{1}{2} i\hbar\mathscr{P})$ change their sign on exchange of factors. On the other hand an odd power of i is imaginary. Therefore one has

$$\overline{A * B} = \bar{B} * \bar{A} \tag{6}$$

where $\overline{}$ denotes complex conjugation. Thus, complex conjugation is an adjoining operation with respect to the $*$ product. The product and complex conjugation form the basic elements to define the structure of a star algebra on the set of functions on $\mathbb{R}^{2n}$. One could investigate an adequate norm and restrictions on the functions in order to obtain a C^* algebra. However, we are more interested in obtaining the algebra of quantum mechanics, the corresponding dynamics, and trace operation. Indeed the commutators of p's and q's calculated with respect to the $*$ product

$$[q_l, p_k] = \mathrm{i}\hbar\delta_{lk} \qquad [p_l, p_k] = [q_l, q_k] = 0 \tag{7}$$

suggest that functions on the quantum mechanical phase space algebraically behave like quantum theoretical operators on Hilbert space.

Let us consider $\hbar$ a continuously varying parameter rather than a constant. If the functions A, B do not depend on $\hbar$ one has for $\hbar \to 0$

$$A * B \to AB \tag{8}$$

and

$$(1/\mathrm{i}\hbar)[A, B] \to \{A, B\}. \tag{9}$$

In this sense the classical phase space is the limit of the quantum mechanical one. Therefore the phase space formulation is especially suited to study relations between the quantum theory and its classical limit.

3. Quantum mechanics on phase space

Quantum theory is more than a mere algebraic structure. A physical theory should contain rules that tell us how physical objects like states, observables, spectra, dynamics and expectations are represented by mathematical objects. These relations between physical and mathematical objects cannot be deduced from an algebraic structure. However, we will see that the algebra together with the correspondence principle give sufficient hints to permit an adequate guess.

In classical statistical mechanics states are represented by real functions on phase space that fulfil the restrictions

$$\int \rho(q, p)\, \mathrm{d}\Omega = \text{constant} \qquad \mathrm{d}\Omega = \mathrm{d}q_1 \ldots \mathrm{d}q_n\, \mathrm{d}p_1 \ldots \mathrm{d}p_n \tag{10}$$

$$\rho(q, p) \geqslant 0. \tag{11}$$

These conditions are not independent of the algebraic structure of phase space. They have to be compatible with the dynamics, which is formulated with the algebraic structure. So, if one requires that all state functions are real, positive and have the same integral then these properties have to be conserved under time evolution. These restrictions are in fact compatible with the classical dynamics

$$\partial\rho/\partial t = \{H, \rho\} \tag{12}$$

because the change in time of ρ according to (12) corresponds to a measure preserving motion of phase points. The simplest guess for a dynamics on $\langle \mathbb{R}^{2n}, * \rangle$ that for $\hbar \to 0$ goes into the classical (12) is

$$\partial\rho/\partial t = (1/\mathrm{i}\hbar)(H * \rho - \rho * H). \tag{13}$$

This, of course, is nothing but the von Neumann equation. What conditions can we put on the functions ρ assuming the $*$ dynamics, equation (13)? Because of (6) we can still assume ρ to be real if H is real, and also restriction (10) is still compatible with (13), which follows from the theorem below.

Theorem 1. Let A and B be two functions on $\mathbb{R}^{2n}$ such that one of them with all its derivatives vanishes at infinity so that all products of the type $(\partial^k A/\partial x_{i_1} \ldots \partial x_{i_k})$ $(\partial^k B/\partial x_{j_1} \ldots \partial x_{j_k})$ vanish at infinity, then $\int A * B \, d\Omega = \int AB \, d\Omega$.

Proof. One has $A * B = AB + \Sigma_{k=1}^{\infty} (1/k!) A(\frac{1}{2}i\hbar\mathscr{P})^k B$. Each term of the sum $\Sigma_{k=1}^{\infty}$ can be written as a sum of Poisson brackets of derivatives of A and B. Liouville's theorem, or simply partial integration, then shows that the term does not contribute to the integral $\int A * B \, d\Omega$.

Theorem 1 does not imply that $\int A * B * C \, d\Omega$ and $\int ABC \, d\Omega$ coincide. However, it follows that factors may be permuted cyclically under the integral

$$\int A * B * C \, d\Omega = \int C * A * B \, d\Omega. \tag{14}$$

Equation (13) and theorem 1 imply that $\int \rho(q, p, t) \, d\Omega$ is constant in time. Classically this constant $\alpha = \int \rho \, d\Omega$ can be chosen arbitrarily. All expectations of observables

$$\langle A \rangle_\rho = \frac{1}{\alpha} \int A\rho \, d\Omega \tag{15}$$

are invariant under a change of α, and the restrictions (10) and (11) do not fix α either. It is one of the most striking aspects of the quantum mechanical phase space that its algebraic structure is suited to fix the value of α. The origin of this normalisation of phase space volume is the quantum version of restriction (11), which we are now going to formulate.

One can show with the aid of counterexamples that restriction (11) is not compatible with $*$ dynamics. This is because (13), unlike its classical analogue, does not describe a motion of phase points. In the classical theory phase points describe pure states $\rho(q, p) = \alpha\delta(q - \xi)\delta(p - \eta)$ and a general state is a (possibly infinite) convex linear combination of such pure δ states. In $*$ mechanics pure states can no longer be associated with phase points. In order to be consistent with restriction (10) the set of functions representing pure states of $*$ mechanics should consist of functions that all have the same integral. One can obtain such a set with the help of algebraical conditions.

Let $\mathscr{A}$ be an algebra with respect to the $*$ product, i.e. a linear space of functions that is closed under the $*$ product, $A, B \in \mathscr{A} \Rightarrow A * B \in \mathscr{A}$. A real function $P \in \mathscr{A}$ will be called pure in $\mathscr{A}$ if for any real $F \in \mathscr{A}$ the equation $P * F = F$ is equivalent to saying that F and P are proportional, $F = cP$, $c \in \mathbb{R}$.

$$\forall F = \bar{F} \in \mathscr{A} \qquad (P * F = F \Leftrightarrow F = cP). \tag{16}$$

The backward implication '$\Leftarrow$' tells us that P is idempotent, $P * P = P$, and the forward implication '$\Rightarrow$' tells us that P is minimal, i.e. it cannot be written as a sum of idempotent real functions.

We call an algebra $\mathscr{A}$ coherent if (i) there exist pure functions in $\mathscr{A}$ and (ii) for any pure functions $P_1, P_2 \in \mathscr{A}$ there exists a pure function $P_3 \in \mathscr{A}$ such that $P_1 * P_3 \neq 0$ and $P_2 * P_3 \neq 0$. In condition (i) one may include a completeness requirement stating that the identity of the algebra $\mathbb{1}_{\mathscr{A}}$ can be resolved by pure functions. However, this is not important for our present purposes.

The following theorem tells us that the pure functions of a coherent algebra are good candidates to represent pure states because they already fulfil condition (10).

Theorem 2. If $\mathscr{A}$ is a coherent algebra whose pure functions fulfil the condition of theorem 1 then all pure functions in $\mathscr{A}$ have the same integral $\int P \, d\Omega = \text{constant}$.

Proof. Let $P_1, P_2 \in \mathscr{A}$ be two arbitrary pure functions. We choose a pure function $P_3 \in \mathscr{A}$ such that $P_1 * P_3 \neq 0$ and $P_2 * P_3 \neq 0$. The function $P_1 * P_3 * P_1$ is a real element of $\mathscr{A}$ and complies with

$$P_1 * (P_1 * P_3 * P_1) = (P_1 * P_3 * P_1) \tag{17}$$

where we used that pure functions are idempotent. According to the definition of pure functions this implies

$$P_1 * P_3 * P_1 = c_1 P_1. \tag{18}$$

In the same way one gets

$$P_3 * P_1 * P_3 = c_3 P_3. \tag{19}$$

One can combine (18) and (19) as follows

$$c_3 P_1 * P_3 = P_1 * P_3 * P_1 * P_3 = c_1 P_1 * P_3 \tag{20}$$

which means $c_3 = c_1$. As P_3 is idempotent one gets using theorem 1, (6), and $P_1 * P_3 \neq 0$ the following inequality

$$\int P_1 * P_3 * P_1 \, d\Omega = \int (P_1 * P_3)(P_3 * P_1) \, d\Omega = \int |P_1 * P_3|^2 \, d\Omega > 0 \tag{21}$$

and thus $c_1 \neq 0$. Then, applying theorem 1 and idempotence one gets

$$\begin{aligned} \int P_1 \, d\Omega &= \frac{1}{c_1} \int P_1 * P_3 * P_1 \, d\Omega = \frac{1}{c_1} \int P_3 * P_1 \, d\Omega \\ &= \frac{1}{c_3} \int P_3 * P_1 * P_3 \, d\Omega = \int P_3 \, d\Omega. \end{aligned} \tag{22}$$

The same argument can be repeated with P_2 so that one finds $\int P_1 \, d\Omega = \int P_3 \, d\Omega = \int P_2 \, d\Omega$.

Let us assume we have a coherent algebra $\mathscr{A}$ as required in theorem 2, and suppose that the Hamiltonian H of (13) is a real element of $\mathscr{A}$. The solution of (13) has the form

$$\rho(t) = U(t) * \rho(0) * \overline{U(t)} \tag{23}$$

where U is a unitary function in $\mathscr{A}$, i.e. a function such that

$$U * \bar{U} = \bar{U} * U = \mathbb{1}_{\mathscr{A}} \tag{24}$$

where $\mathbb{1}_{\mathscr{A}}$ is the unit element in $\mathscr{A}$. It is readily seen that $U * P * \bar{U}$ is pure in $\mathscr{A}$ if and only if P is pure, provided U is unitary in $\mathscr{A}$. That means that pure functions of a coherent algebra do not only fulfil condition (10), they also form a set that is invariant under $*$ dynamics.

Therefore we assume that our quantum theory is associated with a coherent algebra $\mathscr{A}$. All states and observables and especially the Hamiltonian H should be elements of $\mathscr{A}$. The pure functions of $\mathscr{A}$ represent the pure states of the system and a general state will be a (possibly infinite) convex linear combination of pure states

$$\rho = \sum_{k=1}^{\infty} c_k P_k \qquad c_k \geqslant 0 \qquad \sum_{k=1}^{\infty} c_k = 1 \tag{25}$$

where P_k are pure functions. The condition $c_k \geqslant 0$ in (25) replaces the classical condition (11).

The postulate that pure functions belong to the class of functions that represent states fixes the normalisation of phase volume

$$\alpha = \int P \, d\Omega \qquad P \text{ any pure function in } \mathscr{A}. \tag{26}$$

Later we will find that the full algebra of (sufficiently well behaving) functions on $\mathbb{R}^{2n}$ is a coherent algebra and the normalisation is found to be $\alpha = h^n$.

If P_1, P_2 are pure states and P_3 is a different pure state with $P_1 * P_3 \neq 0$ and $P_2 * P_3 \neq 0$ and $(P_1 + P_2) * P_3 = P_3$ then P_3 is called a coherent superposition of P_1 and P_2. The coherence condition for the algebra is the quantum mechanical superposition principle. There are cases where this principle does not apply, namely in the presence of superselection rules. In such a case one simply has to consider a direct sum of several coherent algebras.

States and dynamics are now formalised. It remains to find a representation of observables, expectations and spectra. In classical mechanics observables are represented by real functions on phase space. Let us take this over to $*$ mechanics with the requirement that the corresponding functions are elements of the algebra $\mathscr{A}$. The fact that restriction (10) could be taken over to quantum theory suggests that the expectation, which classically is an integral of the product of state and observable, also in quantum theory is an integral of a product of A and ρ:

$$\langle A \rangle_\rho = \frac{1}{\alpha} \int A * \rho \, d\Omega. \tag{27}$$

According to theorem 1 we may even drop the star and use the classical product, so that $\langle A \rangle_\rho$ precisely takes the classical form of (15), however, α is no longer arbitrary (equation (26)).

The spectrum of an observable is the set of experimental outcomes which can show up in a single experiment with arbitrary state. Classically the spectrum coincides with the range of the function A that represents the observable

$$Sp_{\text{class.}} A = \{A(q, p) \mid (q, p) \in \mathbb{R}^{2n}\}. \tag{28}$$

This set can also be described in the following way. A real number λ is an element of $Sp_{\text{class.}} A$ if and only if one can find a sequence of (classical) states $E_n^{(\lambda)}$ so that

$$\int (A - \lambda)^2 E_n^{(\lambda)} \, d\Omega \to 0 \qquad \text{if } n \to \infty. \tag{29}$$

A natural extrapolation of this definition to quantum theory would be to replace products by star products: a real number λ is an element of SpA if and only if one can find a sequence of (quantum) states $E_n^{(\lambda)}$ so that

$$\int (A-\lambda) * (A-\lambda) * E_n^{(\lambda)} \, d\Omega \to 0 \qquad \text{if } n \to \infty. \tag{30}$$

It may happen that for certain values of λ one can find a constant sequence $E_n^{(\lambda)} = P(\lambda)$. In this case λ is said to belong to the discrete spectrum of A and $P(\lambda)$ is an eigenstate of A. If $\lambda \in SpA$ is not of the discrete spectrum the sequence $E_n^{(\lambda)}$ is said to represent a generalised eigenstate of A.

In an eigenstate of A the observable A has zero variance

$$(\langle A * A\rangle_\rho - \langle A\rangle_\rho^2)^{1/2} = 0 \qquad A * \rho = \lambda\rho. \tag{31}$$

For example the function

$$\rho(q, p) = 2\exp\left(-\frac{p^2/m\omega + m\omega q^2}{\hbar}\right) \tag{32}$$

is pure in the full algebra on $\mathbb{R}^2$ and it is an eigenstate of the observable

$$H = p^2/2m + \tfrac{1}{2}m\omega^2 q^2. \tag{33}$$

One has

$$H * \rho = \tfrac{1}{2}\hbar\omega\rho. \tag{34}$$

The sequence of pure states

$$\rho_n^{(\lambda)} = 2\exp\left(-\frac{p^2/nm\omega + nm\omega(q-\lambda)^2}{\hbar}\right) \qquad n = 1, 2, \ldots \tag{35}$$

represents a generalised eigenstate of q:

$$q * \{\rho_n^{(\lambda)}\}_n = \lambda\{\rho_n^{(\lambda)}\}_n. \tag{36}$$

In the following section we will discuss the connection of the phase space formalism with the usual Hilbert-space formulation of quantum mechanics. It will turn out that functions on phase space correspond to operators on Hilbert space. So for example ρ corresponds to the density operator $\hat{\rho}$. The reader accustomed with Hilbert-space quantum mechanics might thus be surprised about the form of eigen equations in phase space. Note, however, that the eigen equation

$$\hat{H}|\psi\rangle = E|\psi\rangle \tag{37}$$

obviously remains unchanged by multiplying it with $\langle\psi|$

$$\hat{H}|\psi\rangle\langle\psi| = E|\psi\rangle\langle\psi| \tag{38}$$

which then may be written as

$$\hat{H}\hat{\rho} = E\hat{\rho}, \tag{39}$$

which is an eigen equation for the density operator rather than for a Hilbert space vector.

4. Coherent algebras on $\mathbb{R}^{2n}$ and Hilbert space quantum mechanics

Theorem 3. Let $\psi = \psi(x_1, \ldots x_n)$ be a complex distribution on $\mathbb{R}^n$ with $\int |\psi|^2 \, dx_1 \ldots dx_n = 1$. The distribution

$$P_\psi(q_1, \ldots, q_n, p_1, \ldots, p_n) = 2^n \int_{-\infty}^{+\infty} \bar{\psi}(q-y)\psi(q+y)\exp\left(\frac{2}{i\hbar}\sum_l p_l y_l\right) dy_1 \ldots dy_n \tag{40}$$

is pure in the full algebra $\mathscr{D}$ of distributions on $\mathscr{R}^{2n}$.

The proof of theorem 3 is facilitated by the following formula

$$A(q)\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l y_l\right) * B(q)\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l x_l\right) = A(q+x)B(q-y)\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l(x_l+y_l)\right). \tag{41}$$

Proof of theorem 3. P_ψ is real because $f(y)=\bar{\psi}(q-y)\psi(q+y)$ fulfills the condition $\overline{f(y)}=f(-y)$. We show that P_ψ is idempotent. Equation (41) gives

$$P_\psi * P_\psi = 4^n \int \bar{\psi}(q-y+x)\psi(q+y+x)\bar{\psi}(q-x-y)\psi(q+x-y)$$
$$\times\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l(x_l+y_l)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_n. \tag{42}$$

Introducing new variables of integration $t=x+y$, $t'=x-y$ and using $\int|\psi|^2\,\mathrm{d}t'_1\ldots\mathrm{d}t'_n=1$ gives $P_\psi * P_\psi = P_\psi$. It remains to show that $P_\psi * F = F$ implies $P=cF$ for any real F. Let F with $F=\bar{F}$ and $P_\psi * F = F$ be given. We write

$$F(q,p)=\int f(q,x)\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l x_l\right)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{43}$$

Equation (41) gives

$$P_\psi * F = 2^n\int \bar{\psi}(q-y+x)\psi(q+y+x)f(q-y,x)$$
$$\times\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l(x_l+y_l)\right)\mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}y_1\ldots\mathrm{d}y_n$$
$$=\int \bar{\psi}(q+t')\psi(q+t)f\left(q-\frac{t-t'}{2},\frac{t+t'}{2}\right)$$
$$\times\exp\left(\frac{2}{\mathrm{i}\hbar}\sum p_l t_l\right)\mathrm{d}t_1\ldots\mathrm{d}t_n\,\mathrm{d}t'_1\ldots\mathrm{d}t'_n. \tag{44}$$

Then $P_\psi * F = F$ gives together with Fourier's theorem

$$f(q,t)=\psi(q+t)\int\bar{\psi}(q+t')f\left(q-\frac{t-t'}{2},\frac{t+t'}{2}\right)\mathrm{d}t'_1\ldots\mathrm{d}t'_n. \tag{45}$$

The integral is invariant under the transformation $q\to q+t''$, $t\to t+t''$. Thus $f(q,t)$ has the form

$$f(q,t)=\psi(q+t)J(q-t). \tag{46}$$

The condition that F is real implies $\overline{f(q,t)}=f(q,-t)$, which means

$$\bar{\psi}(q+t)\bar{J}(q-t)=\psi(q-t)J(q+t). \tag{47}$$

Let y be a vector such that $\psi(y)\neq 0$. Now choose q and t so that $q-t=y$ and $q+t=z$

$$J(z)=(\bar{J}(y)/\psi(y))\bar{\psi}(z)=\text{constant}\times\bar{\psi}(z). \tag{48}$$

This means $F=2^{-n}\times\text{constant}\times P_\psi$.

Up to a factor P_ψ is a Wigner function [1]. If $\psi_1(x_1, \ldots, x_n)$, $\psi_2(x_1, \ldots, x_n)$ are two functions with $\int |\psi_1|^2 \, \mathrm{d}x_1 \ldots \mathrm{d}x_n = 1$, $\int |\psi_2|^2 \, \mathrm{d}x_1 \ldots \mathrm{d}x_n = 1$ the integral of $P_{\psi_1} * P_{\psi_2}$ is

$$\begin{aligned}\int P_{\psi_1} * P_{\psi_2} \, \mathrm{d}\Omega & \\ &= \int P_{\psi_1} P_{\psi_2} \, \mathrm{d}\Omega \\ &= 4^n \int \bar{\psi}_1(q-y)\psi_1(q+y)\bar{\psi}_2(q-x)\psi_2(q+x) \\ &\quad \times \exp\left(\frac{2}{\mathrm{i}\hbar} \sum p_l(x_l+y_l)\right) \mathrm{d}x_1 \ldots \mathrm{d}y_n \, \mathrm{d}\Omega \\ &= h^n \left| \int \bar{\psi}_1(q)\psi_2(q) \, \mathrm{d}q_1 \ldots \mathrm{d}q_n \right|^2. \end{aligned} \tag{49}$$

This equation determines the normalisation constant α. Choosing $\psi_1 = \psi_2$ one gets

$$\alpha = h^n. \tag{50}$$

Moreover (49) shows that the set of pure functions of theorem 3 fulfills the coherence condition. In order to show that the full algebra $\mathscr{D}$ of (well behaving) distributions is coherent one would have to demonstrate that all pure functions are of the form given by theorem 3. However this is clear from the following obvious connection with a Hilbert space structure.

If we take ψ as the wavefunction of Hilbert space quantum mechanics we can write P_ψ as

$$P_\psi = 2^n \int \langle q+y \,|\, \psi\rangle\langle\psi \,|\, q-y\rangle \exp\left(\frac{2}{\mathrm{i}\hbar} \sum_l p_l y_l\right) \mathrm{d}y_1 \ldots \mathrm{d}y_n. \tag{51}$$

By linear superposition of these equations with different $|\psi\rangle\langle\psi|$'s one gets the state functions ρ that correspond to an arbitrary density operator $\hat{\rho}$. The resulting correspondence between operators and functions on $\mathbb{R}^{2n}$ can then be generalised to a wider class of operators

$$A = 2^n \int \langle q+y|\hat{A}|q-y\rangle \exp\left(\frac{2}{\mathrm{i}\hbar} \sum_l p_l y_l\right) \mathrm{d}y_1 \ldots \mathrm{d}y_n. \tag{52}$$

The correspondence $W\colon \hat{A} \mapsto A$ is the Wigner mapping. Its inverse $W^{-1}\colon A \mapsto \hat{A}$ is known as the Weyl-Moyal quantisation [3, 7]. One can show [4, 8] that

$$\widehat{A * B} = \hat{A}\hat{B}. \tag{53}$$

That means that the Wigner mapping is an algebraic isomorphism. In order to show the equivalence of phase space and Hilbert space quantum mechanics it only remains to show that for any self-adjoint traceless operator $\hat{A}$ one has

$$\operatorname{Tr} \hat{A} = \frac{1}{h^n} \int A \, \mathrm{d}\Omega. \tag{54}$$

We have already shown (54) for operators of the form $\hat{A} = |\psi\rangle\langle\psi|$. As any trace class operator can be written as $\Sigma_n a_n |\psi_n\rangle\langle\psi_n|$, $\Sigma_n |a_n| < \infty$ one gets the general case by linearity and continuity.

Finally we would like to discuss a case where $\mathscr{A}$ is not the full algebra of distributions. Suppose we want to describe a system of N identical Bose particles. Let us denote the $6N$ coordinates by $q_1^1, q_1^2, \ldots p_N^3$ or $\boldsymbol{q}_1, \ldots, \boldsymbol{p}_N$. The phase space function

$$S = \frac{1}{N!} \sum_{\sigma \in S_N} \prod_{i=1}^{N} (\delta_{i\sigma(i)} + h^3(1 - \delta_{i\sigma(i)}) \delta(\boldsymbol{q}_i - \boldsymbol{q}_{\sigma(i)}) \delta(\boldsymbol{p}_i - \boldsymbol{p}_{\sigma(i)})) \tag{55}$$

is real and idempotent, where we define the $*$ product for δ functions with the help of (41), writing $\delta(\boldsymbol{p}_i - \boldsymbol{p}_{\sigma(i)})$ as a Fourier integral. S has the spectrum $\{0, 1\}$. The set of eigenfunctions A with eigenvalue 1, $S * A = A$, is invariant under $*$ multiplication. We take this space $\mathscr{B}$ as the algebra for our quantum theory. The Hamiltonian H should be an element of $\mathscr{B}$. Then H will not coincide with the classical function, which one would use intuitively. H is the projection of the classical Hamiltonian H_{cl} into the algebra $\mathscr{B}$

$$H = S * H_{\text{cl}} * S. \tag{56}$$

Let us investigate the classical limit of the partition function Z

$$Z = \frac{1}{h^{3N}} \int {}_{\mathscr{B}} e^{-\beta H} \, d\Omega \tag{57}$$

${}_{\mathscr{B}} e$ designates the exponential in $\mathscr{B}$. The exponential in an algebra $\mathscr{A}$ is defined by

$${}_{\mathscr{A}} e^X = \mathbb{1}_{\mathscr{A}} + X + \tfrac{1}{2} X * X + \ldots \tag{58}$$

where $\mathbb{1}_{\mathscr{A}}$ is the identity in $\mathscr{A}$. One has $\mathbb{1}_{\mathscr{L}} = 1$ and $\mathbb{1}_{\mathscr{B}} = S$. Using (56), the fact that H_{cl} and S commute and theorem 1 we get

$$\begin{aligned} Z &= \frac{1}{h^{3N}} \int {}_{\mathscr{B}} \exp(-\beta X * H_{\text{cl}} * S) \, d\Omega \\ &= \frac{1}{h^{3N}} \int {}_{\mathscr{L}} e^{-\beta H_{\text{cl}}} S \, d\Omega. \end{aligned} \tag{59}$$

If one now takes the limit $\hbar \to 0$ for the integrand one sees that all terms in S which correspond to permutations σ different from the identity vanish and it remains

$$Z_{\text{cl}} = \frac{1}{N! \, h^{2N}} \int e^{-\beta H_{\text{cl}}} \, d\Omega. \tag{60}$$

The factor $N!$ is often referred to as the reduction of phase volume. Here it stems from the restriction of the algebra $\mathscr{A}$.

5. Polar coordinates as canonical variables

In § 2 we mentioned that the $*$ product in general is not invariant under a change of canonical variables:

$$A * B \neq A *' B \tag{61}$$

with

$$*' = \exp\left[\frac{i\hbar}{2} \sum_l \left(\frac{\overleftarrow{\partial}}{\partial q_l'} \frac{\overrightarrow{\partial}}{\partial p_l'} - \frac{\overleftarrow{\partial}}{\partial p_l'} \frac{\overrightarrow{\partial}}{\partial q_l'}\right)\right].$$

Thus a single classical phase space corresponds to an infinite number of different quantum mechanical phase spaces. Nevertheless, two different quantum mechanical phase spaces may serve to represent the same quantum system in such a way that both representations have the same classical limit in a common classical phase space. If $\mathcal{O}_A$, $\mathcal{O}_B$ and $\mathcal{O}_C$ are observables that in the first space are represented by real functions A, B and C, these observables would be represented in the second space by real functions A', B' and C' such that

$$A=\alpha B * C+\beta C * B \Leftrightarrow A'=\alpha B' *' C'+\beta C' *' B' \tag{62}$$

$$\int A \, \mathrm{d}\Omega=\int A' \, \mathrm{d}\Omega \tag{63}$$

and

$$\begin{matrix} A & & \\ & \searrow & \\ & & A_{\text{classical}} \\ & \nearrow & \\ A' & & \end{matrix} \qquad \text{for } \hbar \to 0. \tag{64}$$

Although the representations are equivalent, the functions A and A' may differ significantly. It can even happen that in the first space the physical algebra contains all well behaving functions whereas in the second space the coherent algebra is formed by a small subalgebra. Polar coordinates give an example for this situation.

Quantum mechanics of a particle that moves in a two-dimensional plane can be described in a phase space defining the $*$ product with respect to the coordinates x, y, p_x, p_y. In this case the coherent algebra is the set $\mathscr{D}$ that we introduced in § 4. Can the same quantum theory be represented if the $*$ product is defined with respect to polar coordinates $r=(x^2+y^2)^{1/2}$, $\varphi=\tan^{-1}(y/x)$ and their conjugated variables p_r, p_φ? In fact it can. However the observables are then represented by distributions that form a small subalgebra $\mathscr{D}_{\text{polar}}$. The angular part φ, p_φ defines a cylindrical phase space. A coherent algebra with respect to the $*$ product defined with φ and p_φ is formed by the functions

$$A(\varphi, p_\varphi)=\sum_{l,k=-\infty}^{+\infty} a_{lk}\chi_{l+k}(p_\varphi)\exp[\mathrm{i}(l-k)\varphi] \tag{65}$$

where a_{lk} are arbitrary complex coefficients and χ_n is defined by

$$\chi_n(p)=\begin{cases} 1 & \text{if } n\hbar \leqslant 2p \leqslant (n+2)\hbar \\ 0 & \text{otherwise.} \end{cases} \tag{66}$$

Using the multiplication law

$$a(p)\exp(-\mathrm{i}qk) * b(p)\exp(-\mathrm{i}qk')=a(p-\tfrac{1}{2}\hbar k')b(p+\tfrac{1}{2}\hbar k)\exp[-\mathrm{i}q(k+k')] \tag{67}$$

and $\chi_n(p)\chi_m(p)=\delta_{nm}\chi_n(p)$ for $n+m$ even, one can show that functions of the type (65) multiply in the following way

$$A(\varphi, p_\varphi) *' B(\varphi, p_\varphi)=\sum_{lk}\left(\sum_j a_{lj}b_{jk}\right)\chi_{l+k}(p_\varphi)\exp[\mathrm{i}(l-k)\varphi] \tag{68}$$

where $*'$ indicates the star product that is defined with coordinates φ, r, p_φ, p_r.

From (68) it is clear that functions of the type (65) form a coherent algebra (a detailed discussion of quantum mechanics on the cylinder is found in [9]).

The radial part r, p_r defines a half-plane, on which a coherent algebra is given by the functions of the following type:

$$A(r, p_r) = \int_{-\infty}^{+\infty} a(r, x) \exp\left(\frac{2}{\mathrm{i}\hbar} p_r x\right) \mathrm{d}x \tag{69}$$

with

$$a(r, x) = 0 \qquad \text{if } |x| > r.$$

The pure functions of this algebra have the form

$$P_\psi(r, p_r) = \int_{-\infty}^{+\infty} \bar{\psi}(r-y)\psi(r+y) \exp\left(\frac{2}{\mathrm{i}\hbar} p_r y\right) \mathrm{d}y \tag{70}$$

with $\psi(y) = 0$ if $y < 0$ and $\int_0^\infty |\psi(y)|^2 \, \mathrm{d}y = 1$. As the coordinate φ loses its meaning for $r = 0$ we cannot simply take the product of these algebras to define $\mathscr{D}_{\text{polar}}$. For $\mathscr{D}_{\text{polar}}$ we take functions of the type

$$A'(\varphi, r, p_\varphi, p_r) = \sum_{l,k} \int a'_{lk}(r, x)\chi_{l+k}(p_\varphi) \exp\left(\frac{2}{\mathrm{i}\hbar} p_r x\right) \exp[\mathrm{i}(l-k)\varphi] \, \mathrm{d}x \tag{71}$$

with

$$a'_{lk}(r, x) = 0 \qquad \text{if } |x| \geqslant r$$

and with special conditions for functions with $a'_{lk}(|x|, x) \neq 0$, which we will give later. A Wigner mapping between $\mathscr{D}_{\text{polar}}$ and operators on the Hilbert space $l^2 \otimes L^2(\mathbb{R}_+)$ is given by

$$A'(\varphi, r, p_\varphi, p_r) = 2\sum_{l,k} \int_{-\infty}^{+\infty} \langle l, r+x|\hat{A}'|k, r-x\rangle \chi_{l+k}(p_\varphi) \exp\left(\frac{2}{\mathrm{i}\hbar} p_r x + \mathrm{i}(l-k)\varphi\right) \mathrm{d}x. \tag{72}$$

Note that the normalisation of states in (72) is such that $\langle l, x | k, y\rangle = \delta_{lk}\delta(x-y)$ and not $\langle l, x | k, y\rangle = (1/x)\delta_{lk}\delta(x-y)$, which one might expect.

Let us construct an isometric isomorphism between $\mathscr{D}$ with the $*$ product and $\mathscr{D}_{\text{polar}}$ with the $*'$ product

$$I\colon A(x, y, p_x, p_y) \mapsto A'(\varphi, r, p_\varphi, p_r) \tag{73}$$

in such a way that not only (62) and (63) are fulfilled but also (64). In order to find this isomorphism we go from a given function $A \in \mathscr{D}$ to an operator $\hat{A}$ over $L^2(\mathbb{R}^2)$ using the Weyl mapping (inverse of (52)). Then we conjugate with a unitary mapping U that takes us from $L^2(\mathbb{R}^2)$ to $l^2 \otimes L^2(\mathbb{R}_+)$ and finally we apply (72) to get $A'(\varphi, r, p_\varphi, p_r)$. The unitary mapping $U\colon L^2(\mathbb{R}^2) \to l^2 \otimes L^2(\mathbb{R}_+)$ we choose such that it maps the wavefunction $\psi(x, y)$ onto the wavefunction

$$\phi(l, r) = \left(\frac{r}{\pi}\right)^{1/2} 2 \int_{-\infty}^{+\infty} \exp[-\mathrm{i}l \tan^{-1}(y/x)] \times \delta(r^2 - x^2 - y^2)\psi(x, y) \, \mathrm{d}x \, \mathrm{d}y \qquad l \in \mathbb{Z}, \qquad r \geqslant 0. \tag{74}$$

Conjugation with U, $\hat{A}' = U\hat{A}U^+$ as well as the Weyl mapping W^{-1} and the Wigner mapping W_{polar} are isometric isomorphisms. Now define I so that the following diagram commutes

$$\begin{array}{ccc} \text{Operators on } L^2(\mathbb{R}^2) & \xrightarrow[\text{conjugation with } U]{} & \text{Operators on } l^2 \otimes L^2(\mathbb{R}_+) \\ \Big\uparrow W^{-1} & & \Big\downarrow W_{\text{polar}} \\ \mathscr{D} & \xrightarrow{\quad I \quad} & \mathscr{D}_{\text{polar}}. \end{array} \tag{75}$$

We see that I is an isometric isomorphism and thus conserves the structure of quantum mechanics. So we obtain an equivalent representation of quantum mechanics of a particle in the plane in $\mathscr{D}_{\text{polar}}$.

It remains to show that the classical limit of this representation is not only equivalent but identical with the orginal limit (i.e. (64) is fulfilled). Direct calculation shows that the observables that in $\mathscr{D}$ are represented by x, y, p_x, p_y in $\mathscr{D}_{\text{polar}}$ are represented by

$$\begin{aligned} x'(\varphi, r, p_\varphi, p_r) &= r \cos \varphi \\ y'(\varphi, r, p_\varphi, p_r) &= r \sin \varphi \\ p'_x(\varphi, r, p_\varphi, p_r) &= p_r \cos \varphi - (1/2r)(P *' \sin \varphi + \sin \varphi *' P) \\ p'_y(\varphi, r, p_\varphi, p_r) &= p_r \sin \varphi + (1/2r)(P *' \cos \varphi + \cos \varphi *' P) \end{aligned} \tag{76}$$

where

$$P = \hbar \sum_{l=-\infty}^{+\infty} l\chi_{2l}(p_\varphi). \tag{77}$$

In the classical limit one has

$$\begin{array}{lll} x \searrow & & \\ & x & \\ x' \nearrow & & \end{array} \qquad \begin{array}{lll} y \searrow & & \\ & y & \\ y' \nearrow & & \end{array} \qquad \begin{array}{lll} p_x \searrow & & \\ & p_x & \\ p'_x \nearrow & & \end{array} \qquad \begin{array}{lll} p_y \searrow & & \\ & p_y. & \\ p'_y \nearrow & & \end{array} \tag{78}$$

As both $*$ and $*'$ go into the classical limit, one concludes from the isomorphism (62) that (64) is also true for $*$ polynomials of x, y, p_x, p_y.

Finally we have to define which of the marginal elements with $a'_{lk}(|x|, x) \neq 0$ belong to $\mathscr{D}_{\text{polar}}$. For physical reasons and in order to have the isomorphism between $\mathscr{D}$ and $\mathscr{D}_{\text{polar}}$ we obviously want x', y', p'_x, p'_y to be elements of $\mathscr{D}_{\text{polar}}$. We may then take $\mathscr{D}_{\text{polar}}$ to be the algebra that is generated by these functions. The canonical variables φ, p_φ and p_r will not be in $\mathscr{D}_{\text{polar}}$.

The unitary mapping U: $L^2(\mathbb{R}^2) \to l^2 \otimes L^2(\mathbb{R}_+)$ is the Hilbert space counterpart of the canonical transformation from cartesian to polar coordinates. In this sense we could say that we found a unitary representation of this transformation. The transformation from cartesian to polar coordinates changes the topology and the range of the coordinates. The standard technique to represent such transformations by unitary operators uses the concept of ambiguity spin [10].

6. Conclusions

Following the ideas of Groenewold [2] we see that functions over the quantum mechanical phase space provide us with the algebra of quantum mechanics. Adding to this algebra the trace and coherence condition, introduced in § 3, we recover the complete features of quantum mechanics. This last step, i.e. the coherence condition and its consequences, are the new features of this work.

A brief comparison with some other approaches is in order. Essentially three lines of thought are usually followed when quantum mechanics on phase space is discussed.

A first line follows Wigner and obtains all relevant quantities from the Wigner map. This obviously yields a complete theory for $\mathbb{R}^{2n}$ with cartesian coordinates. For other manifolds and other coordinates a Wigner map is not available.

Next there is Groenewold's algebraic approach [2], which is well represented in [4, 5]. We consider that the present paper is consistently in this line and actually completes this method with the coherence condition.

Finally Flato and co-workers [6] have handled the problem as one of deformations of Lie algebras. On $\mathbb{R}^{2n}$ they obtain the same results in a more general framework for the $*$ product and they discuss quantum mechanics on other manifolds by embedding in larger spaces. They solve the problem of obtaining quantum mechanics on the cotangent bundle of an n-sphere without introducing anything similar to the coherence condition. This approach differs from ours basically in that the $*$ product on the more complicated manifold is not of the Groenewold type and that quantisation takes place always in $\mathbb{R}^{2m}$ and is then transferred to the smaller manifold.

The general concept of deformed Lie algebras gives rise to a more general $*$ product. Thus one could, e.g., choose

$$\tilde{*} = \exp\left[i\hbar \sum_l \left(\alpha \frac{\overleftarrow{\partial}}{\partial q_l} \frac{\overrightarrow{\partial}}{\partial p_l} - (1-\alpha) \frac{\overleftarrow{\partial}}{\partial p_l} \frac{\overrightarrow{\partial}}{\partial q_l} \right) \right] \qquad 0 \leqslant \alpha \leqslant 1. \tag{79}$$

Indeed most of our arguments will follow through for this case, except for the definition of the trace which can only be given in analogy with the classical case if the Poisson operator is used. The reason for this is that the proof of theorem 1 makes explicit use of the Poisson operator. For other choices of $*$ the definition of the trace becomes cumbersome and the simplest way to define a trace may be to implement the unitary map between the algebras with $*$ and $\tilde{*}$ by a procedure given by Lassner [11].

In this work we have not pursued mathematical rigour. We focused on describing the phase space representation in simple terms accessible to all physicists, trying to convince us that one obtains the same quantum mechanics as in Hilbert space. We have not clearly defined the set of distributions which are admissible for the physical algebra. Subtleties similar to the ones one gets in Hilbert space with unbounded operators should also occur in phase space, giving rise to interesting mathematical questions. These questions have to be answered separately for each $*$ product depending on the chosen coordinate system. The tool for answering them will be the Wigner–Weyl mapping.

It is surprising that relatively simple operators on Hilbert space, such as the permutation of particles or the space inversion, correspond to δ distributions on phase space (cf § 4). However, it turns out that these distributions can be multiplied with the $*$ product. Here a detailed mathematical analysis has to be worked out, whereby interesting new issues could arise for distribution theory.

In § 3 we showed how the general framework of quantum mechanics can be guessed from classical mechanics. In this sense we have a quantisation method. In general, however, we do not have a quantisation in the sense of a mapping of classical observables to quantum observables. For example the representation of the quantum mechanical angular momentum in polar coordinates is a discontinuous function, which does not coincide with the classical angular momentum. Only its classical limit does. Also the Hamiltonian in (13) does not have to be the classical one. So our formalism will in general not be a quantisation scheme in the sense of a quantisation mapping. It is an alternative representation of quantum mechanics, which may for some purposes (e.g. to study the classical limit) be more useful than the Hilbert space representation.

Acknowledgments

We thank J F Plebański for drawing our attention to the subject, and for helping us along the way and M Moshinsky for many useful discussions, which helped clarifying the coordinate dependence of the * product and the relation with Wigner distributions.

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